JOURNAL OF NUMBER THEORY 31, 191–200 (1989)

Indices of Central Embedding Problems and Applications

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Received August 25, 1987; revised April 27, 1988

Every central embedding problem over a local or global number field will become solvable after enlarging the kernel in a certain way. This leads us to define and estimate the index of such an embedding problem. Some applications are discussed

1. WEAK SOLVABILITY OF CENTRAL EMBEDDING PROBLEMS

Let $\mathfrak{G}$ be a profinite group. An embedding problem $(G, A, (\gamma))$ for $\mathfrak{G}$ consists of a finite group G, an epimorphism $\pi: \mathfrak{G} \to G$, a finite G-module A, and a 2-cocycle class $(\gamma) \in H^2(G, A)$ defining a group extension $1 \to A \to G(\gamma) \to G \to 1$. It is called solvable if there is a homomorphism $\psi: \mathfrak{G} \to G(\gamma)$ such that ψ composed with the natural epimorphism $G(\gamma) \to G$ yields π; ψ is also called a solution of the embedding problem. A solution is called proper if it is surjective. In this case $G(\gamma)$ is realized as a quotient group of $\mathfrak{G}$. For a field k of characteristic 0 denote by $G_k = \mathrm{Gal}(\bar{k}/k)$ its absolute Galois group. It is well known that if k is a number field and $\mathfrak{G} = G_k$ then the existence of a solution implies the existence of a proper solution (see Ikeda [I] or Hoechsmann [H, 6.7, p. 101]). We shall often use the following criterion for solvability (see [H, 1.1, p. 82]):

0022-314X/89 $3.00

(1.1) *The embedding problem* $(G, A, (\gamma))$ *for* $\mathfrak{G}$ *is solvable if and only if* (γ) *belongs to the kernel of the inflation map*

$$\inf\colon H^2(G, A) \to H^2(\mathfrak{G}, A).$$

In the following we consider central embedding problems of the form $(G, C_m, (\gamma))$ where G acts trivially on $C_m := (1/m)\mathbb{Z}/\mathbb{Z}$. Such an embedding problem is called weakly solvable if there is a multiple m' of m such that the embedding problem $(G, C_{m'}, (\gamma))$ which is induced by the natural injection $C_m \hookrightarrow C_{m'}$ is solvable. Using (1.1) we easily obtain

(1.2) *The following statements are equivalent*:

(a) $H^3(\mathfrak{G}, \mathbb{Z}) = 0$.

(b) *Every central embedding problem of the form* $(G, C_m, (\gamma))$ *for* $\mathfrak{G}$ *is weakly solvable.*

For a proof rewrite [O2, p. 227, lines 8–23].

It is well known that $H^3(G_k, \mathbb{Z}) = 0$ if k is a local or global number field. In this case we define the index of $(G, C_m, (\gamma))$ to be the smallest multiple m' of m such that the induced embedding problem with kernel $C_{m'}$ is solvable.

2. The Global Index

Let k be a number field and let $(G, C_m, (\gamma))$ be a central embedding problem for G_k. Then its index divides a natural number n if and only if (γ) is in the kernel of the composition of maps

$$H^2(G, C_m) \xrightarrow{\inf} H^2(G_k, C_m) \xrightarrow{j^*_{n,m}} H^2(G_k, C_n),$$

the last map being induced by the natural injection $C_m \hookrightarrow C_n$. In order to estimate the index by a local–global method we proceed as follows: Let $X(k; m)$ denote the kernel of the localization map

$$H^2(G_k, C_m) \to \prod_{\mathfrak{v}} H^2(G_{k_\mathfrak{v}}, C_m).$$

Using the global duality theorem (cf. Poitou [P]), we see that there is a canonical isomorphism

$$X(k; m) \cong \left(\bigcap_{\mathfrak{v}} k^\times \cap k_\mathfrak{v}^{\times m}\right) \Big/ k^{\times m};$$

the last group is well known to have order dividing 2 (see Artin and Tate [A-T, p. 93 ff]). $X(k;m)$ is contained in the kernel of the map

$$j^*_{2m,m}\colon H^2(G_k, C_m) \to H^2(G_k, C_{2m})$$

because on $X(k;m)$ it dualizes the natural map

$$\left(\bigcap_{\mathfrak{v}} k^* \cap k_{\mathfrak{v}}^{*2m}\right)\Big/ k^{*2m} \to \left(\bigcap_{\mathfrak{v}} k^* \cap k_{\mathfrak{v}}^{*m}\right)\Big/ k^{*m}$$

which is trivial. This will show

(2.1) *The index of a central embedding problem* $(G, C_m, (\gamma))$ *is equal to either the l.c.m. of the indices of all the corresponding local embedding problems* $(G_{\mathfrak{v}}, C_m, (\gamma_{\mathfrak{v}}))$ *or its multiple by the order of* $X(k;m)$.

Proof. Using (1.1) we see that if $X(k;m)=0$ a central embedding problem $(G, C_m, (\gamma))$ is solvable if and only if all the corresponding local embedding problems are solvable. If $X(k;m)\neq 0$ the same holds for the induced embedding problem $(G, C_{2m}, (\gamma))$. Now the assertion is obvious.

Remark. It is well known that $H^3(G_k(S), \mathbb{Z})=0$ if Leopoldt's conjecture holds for the pair (k, p) (see, e.g., Haberland [HA, 4.4, p. 46 ff]). In this case we define the S-index of $(G, C_{p^\mu}, (\gamma))$ to be the smallest p-power $m' = p^{\mu'}$, $\mu \leqslant \mu'$, such that the induced embedding problem with kernel $C_{m'}$ has a solution which is unramified outside S. According to Neukirch [N, (8.1), p. 102], in the case where $(G, C_{m'}, (\gamma))$ is solvable, it has a solution which is unramified outside S if and only if $U^S(m') \subset k^{\times m'}$ where for every natural number n

$$U^S(n) := \{a \in k^\times \mid (a) = \mathfrak{a}^n \text{ for some ideal } \mathfrak{a} \text{ of } k,$$
$$\text{and } a \in k_{\mathfrak{v}}^{\times n} \text{ for all } \mathfrak{v} \in S\}.$$

This implies: If $k=\mathbb{Q}$, then the S-index of every central embedding problem for $G_{\mathbb{Q}}(S)$ with kernel C_{p^μ} is equal to its index. Similarly, if p is a regular prime and if $k=\mathbb{Q}(\zeta_{p^\nu})$ for some ν where ζ_{p^ν} is a primitive p^νth root of 1, then the class number of k is prime to p (see Iwasawa [IW]), which shows that the S-index coincides with the index.

So it seems natural to ask: Which number fields k do have the property that, given p and S, the S-index of every central embedding problem for $G_k(S)$ with kernel C_{p^μ} is equal to its index? We do not know the answer and leave this as an open problem.

In the following we estimate the indices of central embedding problems and give some applications.

3. The Local Index

In this section let K/k be a finite Galois extension of local number fields with group $G = \mathrm{Gal}(K/k)$. At first we estimate the index of a central embedding problem $(G, C_m, (\gamma))$ for a cocycle class (γ) belonging to the kernel of the homomorphism

$$i_m^*\colon H^2(G, C_m) \to H^2(G, \mathbb{Q}/\mathbb{Z})$$

which is induced by the natural map $i_m\colon C_m = (1/m)\mathbb{Z}/\mathbb{Z} \hookrightarrow \mathbb{Q}/\mathbb{Z}$: It is sufficient to consider only powers of a fixed prime p, $m = p^e$. Let $\mu_m(k)$ (resp. $\mu_\infty(k)$) denote the group of roots of unity of order dividing m (resp. of order any power of p) contained in k. Denote by $NK^\times$ the image of the norm map of K/k. For a given m, put

$$m^* := \begin{cases} m & \text{if } \mu_m(k) \subset NK^\times, \\ m \cdot [\mu_\infty(k)\colon \mu_\infty(k) \cap NK^\times] & \text{otherwise.} \end{cases}$$

Then we have

(3.1) *The index of a central embedding problem for a class in* Ker i_m^* *is a divisor of* m^*.

Proof. Let $\pi_m\colon \mathbb{Q}/\mathbb{Z} \to \mathbb{Q}/\mathbb{Z}$ be the homomorphism defined by $\pi_m(x) = m \cdot x$, $x \in \mathbb{Q}/\mathbb{Z}$. Then from the short exact sequence

$$0 \to C_m \xrightarrow{i_m} \mathbb{Q}/\mathbb{Z} \xrightarrow{\pi_m} \mathbb{Q}/\mathbb{Z} \to 0$$

we obtain the exact sequence

$$\mathrm{Hom}(G_k, \mathbb{Q}/\mathbb{Z}) \xrightarrow{\pi_m^*} \mathrm{Hom}(G_k, \mathbb{Q}/\mathbb{Z}) \to H^2(G_k, C_m) \xrightarrow{i_m^*} H^2(G_k, \mathbb{Q}/\mathbb{Z}) = 0.$$

By local class field theory we see that the cokernel of π_m^* is isomorphic to the dual group $\mathrm{Hom}(\mu_m(k), \mathbb{Q}/\mathbb{Z})$ of $\mu_m(k)$, dualizing the exact sequence

$$1 \to \mu_m(k) \to k^\times \xrightarrow{\pi_m} k^\times,$$

where $\pi_m(x) = x^m$, $x \in k^\times$. Hence we have

(3.2) $H^2(G_k, C_m)$ *is canonically isomorphic to the dual group of* $\mu_m(k)$.

If we apply a similar argument to G instead of G_k then we obtain an expression for the kernel of the map $i_m^*\colon H^2(G, C_m) \to H^2(G, \mathbb{Q}/\mathbb{Z})$. It is easy to determine the image of Ker i_m^* by the inflation map $\mathrm{inf}_m\colon H^2(G, C_m) \to H^2(G_k, C_m)$ in a straightforward way. Then we have

(3.3) *The subgroup* $\inf_m(\operatorname{Ker} i_m^*)$ *of* $H^2(G_k, C_m)$ *is canonically isomorphic to* $\operatorname{Hom}(\mu_m(k)/\mu_m(k) \cap NK^\times, \mathbb{Q}/\mathbb{Z})$.

For a multiple $m' = p^{e'}$, $e' \geqslant e$, of $m = p^e$, it is also easy to see that the homomorphism

$$j^*_{m',m}: H^2(G_k, C_m) \to H^2(G_k, C_{m'})$$

corresponds to the dual of the map $\pi_{m',m}: \mu_{m'}(k) \to \mu_m(k)$, $\pi_{m',m}(x) = x^{m'/m}$, $x \in \mu_{m'}(k)$. This proves (3.1).

Now we treat embedding problems for classes $(\gamma) \in H^2(G, C_m)$ such that $i_m^*(\gamma) \neq 0$. Denote the order of $i_m^*(\gamma)$ by m_0; this is a divisor of m. Put

$$v = v(K/k) := |\mu_\infty(k) \cap NK^\times|$$

and

$$m' := \text{l.c.m.}\{m_0 \cdot v, m\} = m_1 \cdot m_0 \cdot v.$$

Then there exists a cocycle $\xi \in Z^2(G, C_{m'})$ such that the embedding problem $(G, C_{m'}, (\xi))$ is solvable, i.e., $\inf_{m'}(\xi) = 0$, and such that $i^*_{m'}(\xi) = i^*_{m'}(j^*_{m',m}(\gamma))$ in $H^2(G, \mathbb{Q}/\mathbb{Z})$ (see [M-O, Theorems 1 and 2]). Since $\inf_{m'}(j^*_{m',m}(\gamma) - (\xi)) = \inf_{m'}(j^*_{m',m}(\gamma))$, the indices of the embedding problems for the two classes, $j^*_{m',m}(\gamma) - (\xi)$ and $j^*_{m',m}(\gamma)$ coincide. It is clear by definition that the index for (γ) divides that for $j^*_{m',m}(\gamma)$. Therefore we can apply (3.1) to $j^*_{m',m}(\gamma) - (\xi) \in \operatorname{Ker} i^*_{m'}$ to estimate the index for (γ). Since m' is a multiple of v, we have $\mu_{m'}(k) \supset \mu_\infty(k) \cap NK^\times$, and $(m')^* = m_1 m_0 . |\mu_\infty(k)|$. Thus we have shown

(3.4) *The index of* $(G, C_m, (\gamma))$ *for* $(\gamma) \in H^2(G, C_m) - \operatorname{Ker} i_m^*$ *divides* $m_1 m_0 . |\mu_\infty(k)|$.

4. A Universal Bound for the Global Index

Let K/k be a finite Galois extension of global number fields with Galois group $G = \operatorname{Gal}(K/k)$. Our results (2.1) and (3.4) yield an estimate for the index of any central embedding problem $(G, C_m, (\gamma))$ which depends only on k, on the set of primes of k which are ramified in K/k, and on m. Therefore we have

(4.1) *Let Σ be a finite set of places of the global number field k and let m be a positive integer. Then the indices of all central embedding problems of the form* $(G, C_m, (\gamma))$ *with* $G = \operatorname{Gal}(K/k)$ *and K/k unramified outside Σ divide a number which depends only on k, Σ, and m.*

5. Abundant Central Extensions

Let K/k be a finite Galois extension of global number fields with $G = \mathrm{Gal}(K/k)$ and let k_{ab} (resp. M) be the maximal abelian extension of k (resp. the maximal central extension of K/k) in the algebraic closure of k. Then $M/K \cdot k_{ab}$ is a finite Galois extension with Galois group isomorphic to the dual of $H^2(G, \mathbb{Q}/\mathbb{Z})$ (see, e.g., [M1]). A finite central extension L of K/k is called abundant if and only if $\mathrm{Gal}(L/L \cap K \cdot k_{ab})$ is isomorphic to the dual of $H^2(G, \mathbb{Q}/\mathbb{Z})$. Using the Hochschild–Serre exact sequence it is easy to see that L is abundant if and only if the inflation map $H^2(G, \mathbb{Q}/\mathbb{Z}) \to H^2(\mathrm{Gal}(L/k), \mathbb{Q}/\mathbb{Z})$ is trivial (see, e.g., [M1]). For $(f) \in H^2(G, \mathbb{Q}/\mathbb{Z})$ denote by $m((f))$ the smallest $\exp(\mathrm{Gal}(E/K))$ where E runs over all finite central extensions of K/k such that (f) belongs to the kernel of the inflation map $H^2(G, \mathbb{Q}/\mathbb{Z}) \to H^2(\mathrm{Gal}(E/k), \mathbb{Q}/\mathbb{Z})$. If the order of (f) is m then we can choose a cocycle $\gamma \in Z^2(G, C_m)$ representing the class (f), and $m((f))$ divides the index of the central embedding problem (G, C_m, γ) and our results (2.1) and (3.4) give an estimate for $m((f))$. This also yields an estimate for the smallest $\exp(\mathrm{Gal}(L/K))$ where L runs over all abundant central extensions of K/k because it is equal to the l.c.m. of all $m((f))$, $(f) \in H^2(G, \mathbb{Q}/\mathbb{Z})$.

Remarks. (a) It is obvious that the estimate for $m((f))$ depends on the choice of γ.

(b) The estimates obtained by (2.1) and (3.4) are better than those given in [M2, Theorem 8], because $[\mu_\infty(k) : \mu_\infty(k) \cap NK^\times]$, which was defined in the previous section, divides the exponent of $G_\mathfrak{v}/[G_\mathfrak{v}, G_\mathfrak{v}]$, as is seen by local class field theory, and because the product of the exponents of G and of $H^2(G, \mathbb{Q}/\mathbb{Z})$ divides the order $|G|$ (see, e.g., Huppert [HU, V, 24.5(b), p. 640]).

6. Embedding Problems Associated with the Obstruction of the Hasse Norm Principle

Let K/k be a finite Galois extension of number fields with Galois group $G = \mathrm{Gal}(K/k)$, and let $\mathscr{K} = \mathscr{K}(K/k)$ denote the obstruction to the Hasse norm principle; i.e., $\mathscr{K}$ is the kernel of the map of Tate cohomology groups

$$H^0(G, K^\times) \to H^0(G, K^\times_{\mathbb{A}}),$$

where $K^\times_{\mathbb{A}}$ is the idele group of K. As observed by Tate (see [T, p. 198]), $\mathscr{K}$ is dual to the kernel $\mathscr{H} = \mathscr{H}(K/k)$ of the localization map

$$H^2(G, \mathbb{Q}/\mathbb{Z}) \to \coprod_{\mathfrak{v}} H^2(G_\mathfrak{v}, \mathbb{Q}/\mathbb{Z}).$$

Furthermore, using the duality theorem, it was shown (see, e.g., [O1]), that a finite central extension L of K/k is defined by $\mathscr{H}$; i.e., $\mathscr{H}$ is contained in the kernel of the inflation map

$$H^2(G, \mathbb{Q}/\mathbb{Z}) \to H^2(\mathrm{Gal}(L/k), \mathbb{Q}/\mathbb{Z}),$$

if and only if the following is true:

(6.1) *An element $x \in k^\times$ which is a norm locally everywhere in L/k is a global norm in K/k.*

We define the index of the number knot $\mathscr{K}(K/k)$ of K/k as the number

$$m(\mathscr{K}) = m(\mathscr{K}(K/k)) := \mathrm{Min}\{\exp \mathrm{Gal}(L/K) \mid L \text{ is defined by } \mathscr{H}(K/k)\}.$$

This is clearly the l.c.m. of all $m(f))$ where (f) runs over all elements in $\mathscr{H}$. Therefore we can estimate the index $m(\mathscr{K})$ by (2.1) and (3.1).

Now suppose that G is abelian. It is well known (see, e.g., [Y], Sect. 2), that in this case $H^2(G, \mathbb{Q}/\mathbb{Z})$ can be identified with the group of symplectic pairings on G, i.e., $H^2(G, \mathbb{Q}/\mathbb{Z}) \simeq (G \wedge G)^\wedge$, by sending a class $(f) \in H^2(G, \mathbb{Q}/\mathbb{Z})$ of order m to the pairing

$$\omega_{(f)}\colon G \times G \to C_m, \quad \omega_{(f)}(x, y) := f(x, y) - f(y, x), \quad x, y \in G.$$

Therefore, if the order of G is odd, every $(f) \in H^2(G, \mathbb{Q}/\mathbb{Z})$ can be represented by the cocycle

$$\gamma\colon G \times G \to C_m, \qquad \gamma(x, y) = \tfrac{1}{2}\omega_{(f)}(x, y), \qquad x, y \in G.$$

If the order of G is even, then we choose a bilinear pairing $\gamma\colon G \times G \to C_m$, such that $\omega_{(f)} = \omega_{(\gamma)}$ and represent (f) by γ. Any such representation of (f) is called a standard representation.

(6.2) *Let K/k be a finite abelian extension of number fields with Galois group $G = \mathrm{Gal}(K/k)$. Suppose that $(f) \in \mathscr{H}$ has order m. Then every standard representation of (f) defines a finite central group extension of G—with kernel C_m if m is odd and with kernel C_{2m} if m is even—which splits locally everywhere.*

Proof. If m is odd, the assertion follows directly from the fact that $(f) \in \mathscr{H}$ and from the definition of the standard representation. If m is even, we choose, for every place $\mathfrak{v}$ of k with an extension $\bar{\mathfrak{v}}$ to K, a function $\alpha_\mathfrak{v}\colon G_\mathfrak{v} \to \mathbb{Q}/\mathbb{Z}$ such that

$$\gamma(x, y) = \alpha_\mathfrak{v}(x) + \alpha_\mathfrak{v}(y) - \alpha_\mathfrak{v}(x \cdot y), \qquad x, y \in G_{\bar{\mathfrak{v}}}$$

and compute

$$2m \cdot \alpha_{\mathfrak{v}}(x) = 2 \cdot [\gamma(x, x) + \gamma(x, x^2) + \cdots + \gamma(x, x^{m-1})]$$
$$= m(m-1) \cdot \gamma(x, x) = 0$$

for all $x \in G_{\bar{\mathfrak{v}}}$.

Combining this with (2.1), we obtain

(6.3) *Let K/k be a finite Galois extension of number fields such that $G = \mathrm{Gal}(K/k)$ is abelian. Then $\mathscr{H}(K/k)$ defines a finite central extension L of K/k such that*

(i) $\exp(\mathrm{Gal}(L/k)) = \exp(G)$ *if* $[K:k]$ *is odd;*

(ii) $\exp(\mathrm{Gal}(L/k)) = 2 \cdot |X(k; 2 \cdot \exp(\mathscr{H}))| \cdot \exp(G)$ *if* $[K:k]$ *is even.*

Note that $|X(k; m)| \leqslant 2$, and $=1$ if $\sqrt{-1} \in K$, for example. We can immediately generalize (6.3) by using Holt's results [HO, Corollary to Theorem 2 and Lemma 2]:

(6.4) *Let K/k be a finite Galois extension of number fields such that every Sylow subgroup of* $\mathrm{Gal}(K/k)$ *is abelian. Then the conclusion of* (6.3) *holds for K/k.*

7. The Norm Exponent in Galois Extensions of Number Fields

Again let K/k be a finite Galois extension of number fields with $G = \mathrm{Gal}(K/k)$. We call the number

$$\lambda = \lambda(K/k) := \exp(k^\times / N_{K/k} K^\times)$$

the norm exponent of K/k. When K/k is a Galois extension of local fields, this and even the quotient group $k^\times/NK^\times$ is well known by local class field theory. In the global case, however, the structure of the quotient group is not clear; and it seems natural to determine $\lambda = \lambda(K/k)$. It is obvious that λ is a multiple of the local norm exponent, $\exp(k_{\mathfrak{v}}^\times / NK_{\bar{\mathfrak{v}}}^\times)$, for every place $\mathfrak{v}$ of k and $\bar{\mathfrak{v}}$ of K over $\mathfrak{v}$. By Tschebotarev's density theorem, therefore, we see

(7.1) *The exponent $\lambda(K/k)$ is a multiple of* $\exp(G)$.

An obvious upper bound for λ is the degree $[K:k] = |G|$. The purpose of this section is to give better upper bounds for λ. These will depend on the obstruction $\mathscr{H}(K/k)$ to the Hasse norm principle for K/k. For instance, using the local reciprocity map, we obtain

(7.2) *If the Hasse norm principle holds for* K/k, *then* $\lambda(K/k)$ *is equal to* $\exp(G)$.

In general we use the fact that there is a finite central extension L of K/k with the following property (see (6.1)):

(7.3) *Every* $x \in k^\times$ *which is a norm locally everywhere in* L/k *is a global norm in* K/k.

We define $\mu = \mu(K/k)$ to be the minimum of all $\exp(\mathrm{Gal}(L/k))$ where L runs over all central extensions of K/k satisfying (7.3), and observe

(7.4) *The norm exponent* $\lambda(K/k)$ *divides* $\mu(K/k)$.

Proof. Choose L satisfying (7.3) such that $\mu(K/k) = \exp(\mathrm{Gal}(L/k))$. Using the Tschebotarev density theorem and local class field theory, we see that μ is the l.c.m. of all local norm exponents $\lambda_{\mathfrak{v}}(K/k) := \exp(k_{\mathfrak{v}}^\times / N_{L_{\tilde{\mathfrak{v}}}/k_{\mathfrak{v}}}(L_{\tilde{\mathfrak{v}}}^\times))$ where $\mathfrak{v}$ runs over all places of k and $\tilde{\mathfrak{v}}$ denotes an extension of $\mathfrak{v}$ to L. The assertion is now obvious because L has the property (7.3).

This estimate together with (6.3) implies

(7.5) *Let* K/k *be a finite Galois extension of number fields such that all of the Sylow subgroups of* Gal(K/k) are abelian. *Then* $\lambda(K/k)$ *is equal to the exponent of the group* Gal(K/k) *if* $[K:k]$ *is odd, and divides, in any case,* $4 \cdot \exp(\mathrm{Gal}(K/k))$.

Proof. Let $G^{(p)}$ be a p-Sylow-subgroup of $G = \mathrm{Gal}(K/k)$. Since the restriction map $H^0(G, K^\times) \to H^0(G^{(p)}, K^\times)$ is injective on the p-part of $H^0(G, K^\times)$, we may assume that G itself is abelian. Using (6.3), we see that there is a finite central extension L of K/k satisfying (7.3) such that $\exp(\mathrm{Gal}(L/k)) = \exp(G)$ if $[K:k]$ is odd, and $\exp(\mathrm{Gal}(L/k)) \mid 4 \cdot \exp(G)$ if $[K:k]$ is even. The assertion is then obvious from (7.4).

Using the transitivity of the norm map we obtain from (7.5) estimates for the norm exponents of solvable extensions in terms of the length and the chief factors of a series of normal subgroups with abelian quotients.

Acknowledgments

This paper was prepared while the first named author was a visiting member of Sonderforschungsbereich 170 in Mathematisches Institut der Universität Göttingen. He is grateful to SFB 170 and to Professor Patterson for giving him this opportunity, and to the staffs for the hospitality he received during his stay there.

Both authors are grateful to the referee for his patience and for some detailed comments.

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